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Heat transfer at the boundary between a porous medium and a homogeneous fluid

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Abstract—The heat transfer conditions that apply at the boundary between a porous medium and a homogeneous fluid are developed as flux jump conditions based on the *non-local form* of the volume averaged thermal energy equations for both the fluid and the solid. These jump conditions take the form of surface transport equations that contain excess surface accumulation, convection, and conduction, in addition to a term representing the *excess surface heat exchange*. It would appear that this latter term controls the manner in which the flux from the porous medium to the homogeneous fluid is distributed between the solid and fluid phases that make up the porous medium.

1. INTRODUCTION

The system under consideration is illustrated in Fig. 1 where we have identified the porous medium as the ω-region and the homogeneous fluid as the η-region. The governing equations and interfacial conditions that describe the heat transfer process in this system are given by

$$(\rho c_p)_\beta \frac{\partial T_\beta}{\partial t} + (\rho c_p)_\beta \nabla \cdot (\mathbf{v}_\beta T_\beta) = \nabla \cdot (k_\beta \nabla T_\beta) \quad \text{in the } \beta\text{-phase} \tag{1}$$

$$\text{B.C.1} \quad T_\beta = T_\sigma \quad \text{at the } \beta\text{-}\sigma \text{ interface} \tag{2}$$

$$\text{B.C.2.} \quad -\mathbf{n}_{\beta\sigma} \cdot k_\beta \nabla T_\beta = -\mathbf{n}_{\beta\sigma} \cdot k_\sigma \nabla T_\sigma \quad \text{at the } \beta\text{-}\sigma \text{ interface} \tag{3}$$

$$(\rho c_p)_\sigma \frac{\partial T_\sigma}{\partial t} = \nabla \cdot (k_\sigma \nabla T_\sigma) \quad \text{in the } \sigma\text{-phase.} \tag{4}$$

The analysis of the fluid mechanical problem is described elsewhere [1–3], and we will make use of those results without discussion.

The fluid flow and heat transfer processes that occur in the system shown in Fig. 1 are comparable to what one finds in packed bed catalytic reactors [4–8], in processes involving transpiration cooling [9–11], in drying processes [12] and in a variety of other technological applications [13–18]. The heat transfer process associated with the *boundary* between the ω and η-regions has been analyzed by Prat [19–21] and by Sahraoui and Kaviany [22–24] in terms of numerical experiments. Both studies were restricted to the *one-equation model* of heat transfer processes, and in this work we will deal primarily with the *two-equation model* [17, 25–31].

When several simplifications are made in the flux jump conditions, one obtains the following two results

Flux conditions at the ω–η boundary

β-phase

$$\mathbf{n}_{\omega\eta} \cdot (\mathbf{K}^*_{\beta\omega} \cdot \nabla \langle T_\beta \rangle^\beta_\omega) = \mathbf{n}_{\omega\eta} \cdot (k_\beta \nabla \langle T_\beta \rangle^\beta_\eta) - h_{\beta\sigma}(\langle T_\beta \rangle^\beta_\omega - \langle T_\sigma \rangle^\sigma_\omega) \tag{5}$$

σ-phase

$$\mathbf{n}_{\omega\eta} \cdot (\mathbf{K}_{\sigma\omega} \cdot \nabla \langle T_\sigma \rangle^\sigma_\omega) = h_{\beta\sigma}(\langle T_\beta \rangle^\beta_\omega - \langle T_\sigma \rangle^\sigma_\omega). \tag{6}$$

Here $\mathbf{K}^*_{\beta\omega}$ represents the thermal dispersion tensor for the β-phase in the ω-region, while $\mathbf{K}_{\sigma\omega}$ represents the effective thermal conductivity tensor for the solid phase. The last term in both these results represents the *excess surface heat exchange* between the β and σ-phases, and it is this term that controls how the flux at the ω–η boundary is distributed between the fluid and solid phases. Equations (5) and (6) are obtained on the basis of several simplifications, and probably the most important of these is the imposition of the

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NOMENCLATURE

a_v — $= A_{\beta\sigma}/\mathscr{V}$, interfacial area per unit volume [m^{-1}]
$A_{\beta\sigma}$ — $= A_{\sigma\beta}$, area of the β–σ interface contained in the averaging volume [m^2]
$\mathbf{b}_{\beta\beta}$ — vector field that maps $\nabla\langle T_\beta\rangle^\beta$ onto $\tilde{T}_\beta$ [m]
$\mathbf{b}_{\beta\sigma}$ — vector field that maps $\nabla\langle T_\sigma\rangle^\sigma$ onto $\tilde{T}_\beta$ [m]
$\mathbf{b}_{\sigma\beta}$ — vector field that maps $\nabla\langle T_\beta\rangle^\beta$ onto $\tilde{T}_\sigma$ [m]
$\mathbf{b}_{\sigma\sigma}$ — vector field that maps $\nabla\langle T_\sigma\rangle^\sigma$ onto $\tilde{T}_\sigma$ [m]
c_p — heat capacity [kcal kg^{-1} K^{-1}]
$(\rho c_p)_\beta$ — volumetric heat capacity of the β-phase [kcal m^{-3} K^{-1}]
$(\rho c_p)_\sigma$ — volumetric heat capacity of the σ-phase [kcal m^{-3} K^{-1}]
$(\rho c_p)_{\beta s}$ — excess β-phase surface heat capacity [kcal m^{-2} K^{-1}]
$(\rho c_p)_{\sigma s}$ — excess σ-phase surface heat capacity [kcal m^{-2} K^{-1}]
$\mathbf{g}$ — gravity vector [m^2 s^{-1}]
h — traditional heat transfer coefficient [kcal m^{-3} s^{-1} K^{-1}]
$h_{\beta\sigma}$ — boundary region heat transfer coefficient [kcal m^{-3} s^{-1} K^{-1}]
k_β — thermal conductivity of the β-phase [kcal m^{-2} s^{-1} K^{-1}]
k_σ — thermal conductivity of the σ-phase [kcal m^{-2} s^{-1} K^{-1}]
$\mathbf{K}^*_{\beta\beta}$ — total β-phase thermal dispersivity tensor associated with $\nabla\langle T_\beta\rangle^\beta$ [kcal m^{-2} s^{-1} K^{-1}]
$\mathbf{K}^*_{\beta\sigma}$ — total β-phase thermal dispersivity tensor associated with $\nabla\langle T_\sigma\rangle^\sigma$ [kcal m^{-2} s^{-1} K^{-1}]
$\mathbf{K}_{\sigma\beta}$ — σ-phase effective thermal conductivity tensor associated with $\nabla\langle T_\beta\rangle^\beta$ [kcal m^{-2} s^{-1} K^{-1}]
$\mathbf{K}_{\sigma\sigma}$ — σ-phase effective thermal conductivity tensor associated with $\nabla\langle T_\sigma\rangle^\sigma$ [kcal m^{-2} s^{-1} K^{-1}]
$\mathbf{K}^*_{\beta\beta\omega}$ — total β-phase thermal dispersivity tensor associated with $\nabla\langle T_\beta\rangle^\beta_\omega$ [kcal m^{-2} s^{-1} K^{-1}]
$\mathbf{K}^*_{\beta\sigma\omega}$ — total β-phase thermal dispersivity tensor associated with $\nabla\langle T_\sigma\rangle^\sigma_\omega$ [kcal m^{-2} s^{-1} K^{-1}]
$\mathbf{K}_{\sigma\beta\omega}$ — σ-phase effective thermal conductivity tensor associated with $\nabla\langle T_\beta\rangle^\beta_\omega$ [kcal m^{-2} s^{-1} K^{-1}]
$\mathbf{K}_{\sigma\sigma\omega}$ — σ-phase effective thermal conductivity tensor associated with $\nabla\langle T_\sigma\rangle^\sigma_\omega$ [kcal m^{-2} s^{-1} K^{-1}]
$\mathbf{n}_{\beta\sigma}$ — $= -\mathbf{n}_{\sigma\beta}$, unit normal vector directed from the β-phase toward the σ-phase
$\mathbf{n}_\omega$ — outwardly directed unit normal vector for the ω-region
$\mathbf{n}_\eta$ — outwardly directed unit normal for the η-region
$\mathbf{n}_{\omega\eta}$ — $= -\mathbf{n}_{\eta\omega}$, unit normal vector directed from the ω-region toward the η-region
$\mathbf{P}$ — $= \mathbf{I} - \mathbf{n}_{\omega\eta}\mathbf{n}_{\omega\eta}$, projection tensor
p_β — pressure in the β-phase [N m^{-2}]
r_0 — radius of the averaging volume [m]
s_β — scalar field that maps $\langle T_\sigma\rangle^\sigma - \langle T_\beta\rangle^\beta$ onto $\tilde{T}_\beta$
s_σ — scalar field that maps $\langle T_\sigma\rangle^\sigma - \langle T_\beta\rangle^\beta$ onto $\tilde{T}_\sigma$
T_β — temperature in the β-phase [K]
$\langle T_\beta\rangle^\beta$ — intrinsic average temperature in the β-phase [K]
$\tilde{T}_\beta$ — $= T_\beta - \langle T_\beta\rangle^\beta$, spatial deviation for the β-phase [K]
$\langle T_\beta\rangle^\beta_\omega$ — intrinsic average temperature in the β-phase determined by the β-phase thermal energy equation valid in the homogeneous ω-region [K]
$\langle T_\beta\rangle^\beta_\eta$ — intrinsic average temperature in the β-phase determined by the β-phase thermal energy equation valid in the homogeneous η-region [K]
T_σ — temperature in the σ-phase [K]
$\langle T_\sigma\rangle^\sigma$ — intrinsic average temperature in the σ-phase [K]
$\tilde{T}_\sigma$ — $= T_\sigma - \langle T_\sigma\rangle^\sigma$, spatial deviation for the σ-phase [K]
$\langle T_\sigma\rangle^\sigma_\omega$ — intrinsic average temperature in the β-phase determined by the σ-phase thermal energy equation valid in the homogeneous ω-region [K]
$\langle T_\beta\rangle^\beta_s$ — intrinsic average temperature of the β-phase at the ω–η boundary [K]
$\langle T_\sigma\rangle^\sigma_s$ — intrinsic average temperature of the σ-phase at the ω–η boundary [K]
$\mathbf{u}_{\beta\beta}$ — a velocity that represents convective-like transport in the β-phase that is associated with $\langle T_\beta\rangle^\beta$ [m s^{-1}]
$\mathbf{u}_{\beta\sigma}$ — a velocity that represents convective-like transport in the β-phase that is associated with $\langle T_\sigma\rangle^\sigma$ [m s^{-1}]
$\mathbf{u}_{\sigma\beta}$ — a velocity that represents convective-like transport in the σ-phase that is associated with $\langle T_\beta\rangle^\beta$ [m s^{-1}]
$\mathbf{u}_{\sigma\sigma}$ — a velocity that represents convective-like transport in the β-phase that is associated with $\langle T_\sigma\rangle^\sigma$ [m s^{-1}]

NOMENCLATURE *(continued)*

$\mathbf{v}_\beta$	velocity in the β-phase [m s^{-1}]
$\langle \mathbf{v}_\beta \rangle$	superficial average velocity for the β-phase [m s^{-1}]
$\langle \mathbf{v}_\beta \rangle^\beta$	$= \varepsilon_\beta \langle \mathbf{v}_\beta \rangle$, intrinsic average velocity for the β-phase [m s^{-1}]
$\tilde{\mathbf{v}}_\beta$	$= \mathbf{v}_\beta - \langle \mathbf{v}_\beta \rangle^\beta$, spatial deviation velocity for the β-phase [m s^{-1}]
$\langle \mathbf{v}_\beta \rangle_\omega$	superficial average velocity for the β-phase determined by equations valid in the homogeneous ω-region [m s^{-1}]
$\langle \mathbf{v}_\beta \rangle_\eta$	superficial average velocity for the β-phase that is determined by equations that are valid in the homogeneous η-region [m s^{-1}]
$\langle \mathbf{v}_\beta \rangle_s$	superficial average velocity of the β-phase at the ω–η boundary [m s^{-1}]
$\mathscr{V}$	averaging volume [m^3]
V_β	volume of the β-phase contained within the averaging volume [m^3]
$\mathscr{V}_\infty$	large-scale volume [m^3]
V_ω	volume of the entire ω-region contained in $\mathscr{V}_\infty$ [m^3]
V_η	volume of the entire η-region contained in $\mathscr{V}_\infty$ [m^3]
$\mathbf{x}$	position vector locating the centroid of the averaging volume [m]
$\mathbf{y}_\beta$	position vector locating points in the β-phase relative to the centroid [m].

Greek symbols

α_β	thermal diffusivity for the β-phase [m^2 s^{-1}]
ε_β	$= V_\beta/\mathscr{V}$, porosity or volume fraction of the β-phase
ε_σ	$= V_\sigma/\mathscr{V}$, volume fraction of the σ-phase
μ_β	viscosity of the β-phase [Ns m^{-2}]
ρ_β	density of the β-phase [kg m^{-3}]
ρ_σ	density of the σ-phase [kg m^{-3}].

Subscripts

β	identifies a quantity associated with the β-phase
σ	identifies a quantity associated with the σ-phase
$\beta\sigma$	identifies a quantity associated with the β–σ interface
ω	identifies a quantity associated with the ω-region
η	identifies a quantity associated with the η-region
$\omega\eta$	identifies a quantity associated with the ω–η region
s	identifies a surface quantity having the units of the associated bulk quantity times length
$\mathbf{x}$	indicates that a quantity is evaluated at the centroid of the averaging volume.

Superscripts

β	indicates an intrinsic volume averaged quantity associated with the β-phase
σ	indicates an intrinsic volume averaged quantity associated with the σ-phase.

condition of *local gradient equilibrium*. This condition suggests that the *gradients* of $\langle T_\beta \rangle^\beta_\omega$ and $\langle T_\sigma \rangle^\sigma_\omega$ can be equated even when the two temperatures are not equal, i.e. $\langle T_\beta \rangle^\beta_\omega \neq \langle T_\sigma \rangle^\sigma_\omega$; however, we must remind the reader that the *constraints* associated with this condition have not yet been obtained.

In order to analyze the heat transfer process that takes place in the system illustrated in Fig. 1, we need to develop volume averaged transport equations that apply within the ω-region and we need to develop jump conditions that apply at the boundary between the ω-region and the η-region. If we make use of *volume averaged equations* in the ω-region and *point equations* in the η-region, we are confronted with a mismatch of length scales that can only be overcome by the use of a variable-sized averaging volume in the neighborhood of the boundary region. This complication can be avoided if volume averaged equations can be used in *both* the ω and η-regions, and it is this approach that we will follow in this paper. This idea is illustrated in Fig. 2 where we have shown averaging volumes in the ω-region (I), in the boundary region (II), and in the η-region (III). We will derive the governing equations that are applicable in the boundary region, and then indicate the special forms that they take in the homogeneous ω and η-regions. By 'homogeneous ω and η-regions', we mean those *portions* of the ω and η-regions that are *not influenced* by the rapid changes in structure that occur in the boundary region.

1.1. *Volume averaging*

The volume averaged forms of the thermal energy equations for homogeneous regions have been discussed in considerable detail elsewhere [25–32]; however, in this particular study we need equations that are valid *everywhere* in the system illustrated in Fig. 1. This means that we must avoid imposing any length-

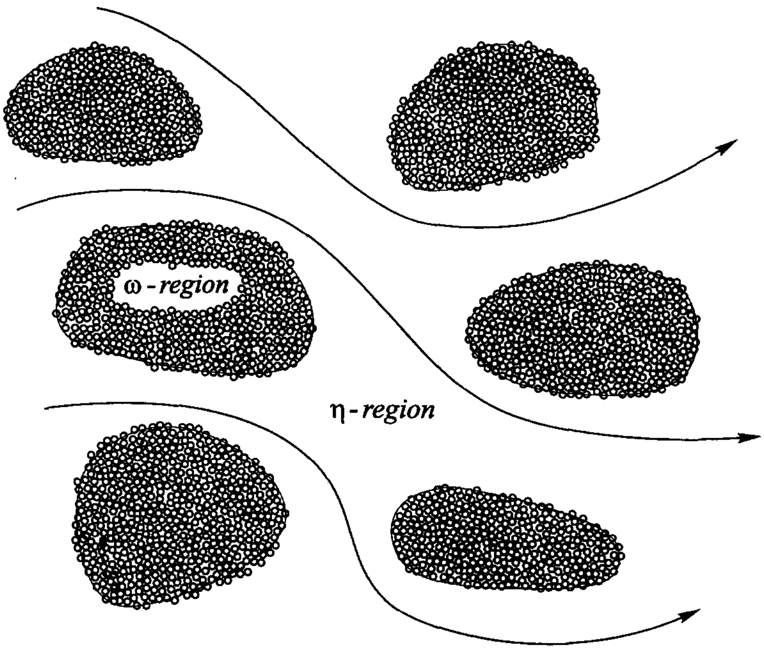

Fig. 1. Convective heat transfer in a system composed of a porous medium and a homogeneous fluid.

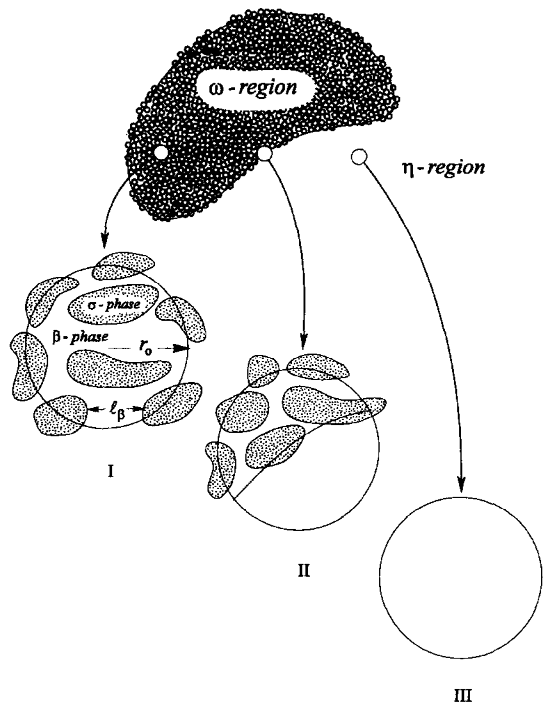

Fig. 2. Averaging volumes.

scale constraints since these will fail in the boundary region (II) illustrated in Fig. 2. In addition to avoiding any length-scale constraints, we must also put forth reasonable representations for the *non-local* transport equations that apply in the boundary region, and this means that the volume averaged forms of equations (1)–(4) must be considered in some detail. After deriving generalized energy transport equations that apply everywhere, we can extract the special forms that describe the heat transfer processes in the homogeneous ω and η-regions.

We begin by defining the superficial volume average of some function ψ_β associated with the β-phase according to

$$\langle \psi_\beta \rangle|_{\mathbf{x}} = \frac{1}{\mathscr{V}} \int_{V_\beta(\mathbf{x})} \psi_\beta(\mathbf{x}+\mathbf{y}_\beta)\, \mathrm{d}V_y. \tag{7}$$

Here $V_\beta(\mathbf{x})$ is the volume of the β-phase contained within the spherical averaging volumes that are shown in Fig. 2, and the position vectors $\mathbf{x}$ and $\mathbf{y}_\beta$ are identified in Fig. 3. There we have indicated that $\mathbf{x}$ represents the position vector locating the centroid of the averaging volume, and that $\mathbf{y}_\beta$ represents the position vector locating points in the β-phase relative to the centroid. Equation (7) clearly indicates that volume averaged quantities are associated with the centroid and that integration is carried out with respect to the components of $\mathbf{y}_\beta$. In general, we will avoid the precise nomenclature used in equation (7) and represent the *superficial average* of ψ_β

$$\langle \psi_\beta \rangle = \frac{1}{\mathscr{V}} \int_{V_\beta} \psi_\beta\, \mathrm{d}V \tag{8}$$

while the *intrinsic average* is expressed in the form

$$\langle \psi_\beta \rangle^\beta = \frac{1}{V_\beta} \int_{V_\beta} \psi_\beta\, \mathrm{d}V. \tag{9}$$

Both these averages will be used in our theoretical development and they are related by

$$\langle \psi_\beta \rangle = \varepsilon_\beta \langle \psi_\beta \rangle^\beta. \tag{10}$$

The porosity ε_β is defined explicitly as

$$\varepsilon_\beta = V_\beta / \mathscr{V} \tag{11}$$

and we note that in the boundary region ε_β undergoes significant changes over a distance equivalent to the radius of the averaging volume, r_0, that is illustrated in Fig. 2.

In terms of the nomenclature illustrated in equation (8), we express the superficial average of equation (1) as

$$\left\langle (\rho c_p)_\beta \frac{\partial T_\beta}{\partial t} + (\rho c_p)_\beta \nabla \cdot (\mathbf{v}_\beta T_\beta) \right\rangle = \langle \nabla \cdot (k_\beta \nabla T_\beta) \rangle. \tag{12}$$

Throughout this development we will ignore variations of the physical properties within the averaging volume, and this allows us to express the first term in equation (12) as

$$\left\langle (\rho c_p)_\beta \frac{\partial T_\beta}{\partial t} \right\rangle = (\rho c_p)_\beta \left\langle \frac{\partial T_\beta}{\partial t} \right\rangle. \tag{13}$$

We consider the σ-phase to be rigid and this allows us to interchange integration and differentiation in order to express the accumulation of thermal energy in the form

$$\left\langle (\rho c_p)_\beta \frac{\partial T_\beta}{\partial t} \right\rangle = (\rho c_p)_\beta \frac{\partial \langle T_\beta \rangle}{\partial t}. \tag{14}$$

Use of the relation between the superficial average and the intrinsic average given by equation (10), and the fact that ε_β is independent of time, leads to the final form for the local volume average of the accumulation term

$$\left\langle (\rho c_p)_\beta \frac{\partial T_\beta}{\partial t} \right\rangle = \varepsilon_\beta (\rho c_p)_\beta \frac{\partial \langle T_\beta \rangle^\beta}{\partial t}. \tag{15}$$

The second term in equation (12) can be expressed as

$$\langle (\rho c_p)_\beta \nabla \cdot (\mathbf{v}_\beta T_\beta) \rangle = (\rho c_p)_\beta \langle \nabla \cdot (\mathbf{v}_\beta T_\beta) \rangle \tag{16}$$

and in order to interchange integration and differentiation we make use of the averaging theorem [33] to express the superficial average of the convective flux in the form

$$(\rho c_p)_\beta \langle \nabla \cdot (\mathbf{v}_\beta T_\beta) \rangle = (\rho c_p)_\beta \left[\nabla \cdot \langle \mathbf{v}_\beta T_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot (\mathbf{v}_\beta T_\beta) \, dA \right]. \tag{17}$$

We assume that the β–σ interface is impermeable so that this result reduces to

$$\langle (\rho c_p)_\beta \nabla \cdot (\mathbf{v}_\beta T_\beta) \rangle = (\rho c_p)_\beta \nabla \cdot \langle \mathbf{v}_\beta T_\beta \rangle \tag{18}$$

and we are ready to move on to the conduction term on the right hand side of equation (12). Use of the averaging theorem leads directly to

$$\langle \nabla \cdot (k_\beta \nabla T_\beta) \rangle = \nabla \cdot \langle k_\beta \nabla T_\beta \rangle + \frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot k_\beta \nabla \, dA \tag{19}$$

and substitution of equations (15), (18) and (19) into equation (12) provides the following form of the superficial averaged thermal energy equation

$$\underbrace{\varepsilon_\beta (\rho c_p)_\beta \frac{\partial \langle T_\beta \rangle^\beta}{\partial t}}_{\text{accumulation}} + \underbrace{(\rho c_p)_\beta \nabla \cdot \langle \mathbf{v}_\beta T_\beta \rangle}_{\text{convection}} = \underbrace{\nabla \cdot \langle k_\beta \nabla T_\beta \rangle}_{\text{conduction}} + \underbrace{\frac{1}{\mathscr{V}} \int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma} \cdot k_\beta \nabla T_\beta \, dA}_{\text{interfacial flux}}. \tag{20}$$

At this point it is important to note that we have imposed *no length-scale constraints* on the volume averaged transport equation, and that the only simplification we have used in deriving this result from equation (1) is the assumption that the variation of physical properties could be ignored within the averaging volume.

The traditional representation of the convective flux is given by [34]

$$\langle \mathbf{v}_\beta T_\beta \rangle = \varepsilon_\beta \langle \mathbf{v}_\beta \rangle^\beta \langle T_\beta \rangle^\beta + \langle \tilde{\mathbf{v}}_\beta \tilde{T}_\beta \rangle \tag{21}$$

in which $\tilde{\mathbf{v}}_\beta$ and $\tilde{T}_\beta$ are the spatial deviation velocity and temperature defined by the following decompositions

$$\mathbf{v}_\beta = \langle \mathbf{v}_\beta \rangle^\beta + \tilde{\mathbf{v}}_\beta \tag{22a}$$

$$T_\beta = \langle T_\beta \rangle^\beta + \tilde{T}_\beta. \tag{22b}$$

Use of equation (21) would require the imposition of length-scale constraints [29], and we need to avoid this in order to develop a transport equation that is valid *within the boundary region* between the ω-region and the η-region. To avoid imposing length-scale constraints, we define an *excess dispersive flux* according to

$$\langle \mathbf{v}_\beta T_\beta \rangle_{ex} = \langle \mathbf{v}_\beta T_\beta \rangle - \varepsilon_\beta \langle \mathbf{v}_\beta \rangle^\beta \langle T_\beta \rangle^\beta - \langle \tilde{\mathbf{v}}_\beta \tilde{T}_\beta \rangle \tag{23}$$

with the idea that

$$\langle \mathbf{v}_\beta T_\beta \rangle_{ex} = 0 \quad \text{in homogeneous regions.} \tag{24}$$

Once again we note that we will use the phrase, 'homogeneous ω-region and homogeneous η-region' to describe those *portions* of the ω and η-regions that are not influenced by the rapid changes in structure that occur in the boundary region.

Use of equation (23) in equation (20) leads to a form that contains the traditional convective and dis-

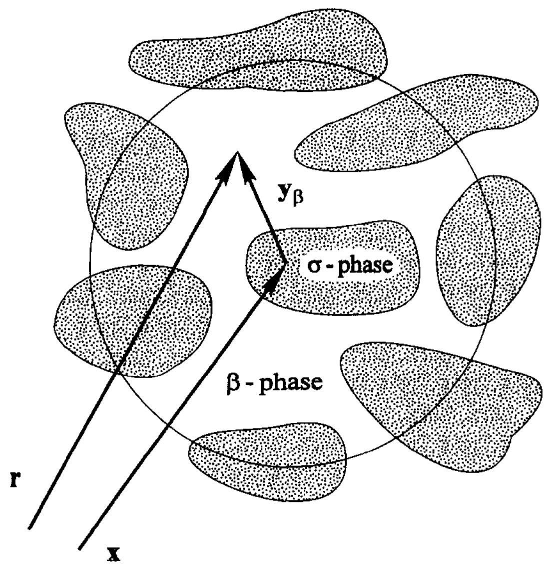

Fig. 3. Position vectors associated with the averaging volume.

persive transport terms in addition to the excess dispersion.

$$\varepsilon_\beta(\rho c_p)_\beta \frac{\partial\langle T_\beta\rangle^\beta}{\partial t} + (\rho c_p)_\beta \nabla\cdot(\varepsilon_\beta\langle \mathbf{v}_\beta\rangle^\beta\langle T_\beta\rangle^\beta) + (\rho c_p)_\beta\nabla\cdot\langle\tilde{\mathbf{v}}_\beta\tilde{T}_\beta\rangle + \underbrace{(\rho c_p)_\beta\nabla\cdot\langle\mathbf{v}_\beta T_\beta\rangle_{ex}}_{\text{non-local dispersion}} = \nabla\cdot\langle k_\beta\nabla T_\beta\rangle + \frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot k_\beta\nabla T_\beta\,\mathrm{d}A. \quad (25)$$

Here we have identified $(\rho c_p)_\beta\nabla\cdot\langle\mathbf{v}_\beta T_\beta\rangle_{ex}$ as a non-local term since it involves, indirectly, values of $\langle T_\beta\rangle^\beta$ that are not associated with the centroid of the averaging volume illustrated in Fig. 3.

Turning our attention to the conductive transport term on the right hand side of equation (25), we ignore variations of k_β within the averaging volume and make use of the averaging theorem to obtain

$$\langle k_\beta\nabla T_\beta\rangle = k_\beta\langle\nabla T_\beta\rangle = k_\beta\left[\nabla\langle T_\beta\rangle + \frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}}\mathbf{n}_{\beta\sigma}T_\beta\,\mathrm{d}A\right]. \quad (26)$$

We then employ equation (10) in order to express this result in terms of the preferred intrinsic average temperature

$$\langle k_\beta\nabla T_\beta\rangle = k_\beta\left[\varepsilon_\beta\nabla\langle T_\beta\rangle^\beta + \langle T_\beta\rangle^\beta\nabla\varepsilon_\beta + \frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}}\mathbf{n}_{\beta\sigma}T_\beta\,\mathrm{d}A\right]. \quad (27)$$

The area integral in this result represents the last obstacle in our route to a volume average transport equation that contains only average quantities and spatial deviations. We attack this area integral by first noting that the averaging theorem provides

$$\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}}\mathbf{n}_{\beta\sigma}\,\mathrm{d}A = -\nabla\varepsilon_\beta \quad (28)$$

and that this allows us to write

$$\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}}\mathbf{n}_{\beta\sigma}\langle T_\beta\rangle^\beta|_{\mathbf{x}}\,\mathrm{d}A = -\langle T_\beta\rangle^\beta\nabla\varepsilon_\beta. \quad (29)$$

Here it is understood that averaged quantities located *outside an integral* are evaluated at the centroid. Use of this result with equation (27) leads to the following expression for the conductive transport

$$\langle k_\beta\nabla T_\beta\rangle = k_\beta\left[\varepsilon_\beta\nabla\langle T_\beta\rangle^\beta + \frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}}\mathbf{n}_{\beta\sigma}(T_\beta - \langle T_\beta\rangle^\beta|_{\mathbf{x}})\,\mathrm{d}A\right]. \quad (30)$$

At this point we can make use of the decomposition given by equation (22b) in order to express this flux in terms of the traditional form [32] plus a *non-local* term

$$\langle k_\beta\nabla T_\beta\rangle = k_\beta\left[\varepsilon_\beta\nabla\langle T_\beta\rangle^\beta + \frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}}\mathbf{n}_{\beta\sigma}\tilde{T}_\beta\,\mathrm{d}A + \underbrace{\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}}\mathbf{n}_{\beta\sigma s}(\langle T_\beta\rangle^\beta - \langle T_\beta\rangle^\beta|_{\mathbf{x}})\,\mathrm{d}A}_{\text{non-local conduction}}\right]. \quad (31)$$

The last term on the right hand side is identified as a non-local term since it involves values of $\langle T_\beta\rangle^\beta$ that are evaluated at points within the averaging volume that are not located at the centroid.

Substitution of equation (31) into equation (25) yields a general form for the β-phase transport equation given by

β-phase

$$\varepsilon_\beta(\rho c_p)_\beta\frac{\partial\langle T_\beta\rangle^\beta}{\partial t} + (\rho c_p)_\beta\nabla\cdot(\varepsilon_\beta\langle\mathbf{v}_\beta\rangle^\beta\langle T_\beta\rangle^\beta) + (\rho c_p)_\beta\nabla\cdot\langle\tilde{\mathbf{v}}_\beta\tilde{T}_\beta\rangle + \underbrace{(\rho c_p)_\beta\nabla\cdot\langle\mathbf{v}_\beta T_\beta\rangle_{ex}}_{\text{non-local dispersion}}$$
$$= \nabla\cdot\left[k_\beta\left(\varepsilon_\beta\nabla\langle T_\beta\rangle^\beta + \frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}}\mathbf{n}_{\beta\sigma}\tilde{T}_\beta\,\mathrm{d}A + \underbrace{\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}}\mathbf{n}_{\beta\sigma}(\langle T_\beta\rangle^\beta - \langle T_\beta\rangle^\beta|_{\mathbf{x}})\,\mathrm{d}A}_{\text{non-local conduction}}\right)\right] + \frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}}\mathbf{n}_{\beta\sigma}\cdot k_\beta\nabla T_\beta\,\mathrm{d}A. \quad (32)$$

The procedure leading to the β-phase transport equation can be repeated for the σ-phase beginning with equation (4), and the result is given by

σ-phase

$$\varepsilon_\sigma(\rho c_p)_\sigma \frac{\partial\langle T_\sigma\rangle^\sigma}{\partial t} = \nabla\cdot\Bigg[k_\sigma\Bigg(\varepsilon_\sigma\nabla\langle T_\sigma\rangle^\sigma + \frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\tilde{T}_\sigma \,\mathrm{d}A + \underbrace{\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}(\langle T_\sigma\rangle^\sigma - \langle T_\sigma\rangle^\sigma|_{\mathbf{x}})\,\mathrm{d}A}_{\text{non-local conduction}}\Bigg)\Bigg] + \frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\cdot k_\sigma\nabla T_\sigma\,\mathrm{d}A. \tag{33}$$

The interfacial flux terms in equations (32) and (33) are equal and opposite, and they will cancel if these two transport equations can be added to obtain a one-equation model. This requires that the condition of local thermal equilibrium be valid [35–37] and we will consider that special case in subsequent paragraphs.

In homogeneous regions, the route to *closed forms* of equations (32) and (33) is reasonably well understood [25–32]; however, we need closed forms that are valid in the boundary region and this requires some judgment that is based on our knowledge of the spatial deviation temperatures. This motivates our use of the decomposition given by equation (22b) in order to express the interfacial flux in terms of both $\langle T_\beta\rangle^\beta$ and $\tilde{T}_\beta$ so that equation (32) takes the form

β-phase

$$\varepsilon_\beta(\rho c_p)_\beta \frac{\partial\langle T_\beta\rangle^\beta}{\partial t} + (\rho c_p)_\beta\nabla\cdot(\varepsilon_\beta\langle \mathbf{v}_\beta\rangle^\beta\langle T_\beta\rangle^\beta) + (\rho c_p)_\beta\nabla\cdot\langle\tilde{\mathbf{v}}_\beta\tilde{T}_\beta\rangle + \underbrace{(\rho c_p)_\beta\nabla\cdot\langle\mathbf{v}_\beta T_\beta\rangle_{\mathrm{ex}}}_{\text{non-local dispersion}}$$
$$= \nabla\cdot\Bigg[k_\beta\Bigg(\varepsilon_\beta\nabla\langle T_\beta\rangle^\beta + \frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\tilde{T}_\beta\,\mathrm{d}A + \underbrace{\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}(\langle T_\beta\rangle^\beta - \langle T_\beta\rangle^\beta|_{\mathbf{x}})\,\mathrm{d}A}_{\text{non-local conduction}}\Bigg)\Bigg] + \underbrace{\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot k_\beta\nabla\langle T_\beta\rangle^\beta\,\mathrm{d}A}_{\text{non-local heat exchange}} + \frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot k_\beta\nabla\tilde{T}_\beta\,\mathrm{d}A. \tag{34}$$

The analogous form of the σ-phase transport equation is given by

σ-phase

$$\varepsilon_\sigma(\rho c_p)_\sigma \frac{\partial\langle T_\sigma\rangle^\sigma}{\partial t} = \nabla\cdot\Bigg[k_\sigma\Bigg(\varepsilon_\sigma\nabla\langle T_\sigma\rangle^\sigma + \frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\tilde{T}_\sigma\,\mathrm{d}A + \underbrace{\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}(\langle T_\sigma\rangle^\sigma - \langle T_\sigma\rangle^\sigma|_{\mathbf{x}})\,\mathrm{d}A}_{\text{non-local conduction}}\Bigg)\Bigg] + \underbrace{\frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\cdot k_\sigma\nabla\langle T_\sigma\rangle^\sigma\,\mathrm{d}A}_{\text{non-local heat exchange}} + \frac{1}{\mathscr{V}}\int_{A_{\sigma\beta}} \mathbf{n}_{\sigma\beta}\cdot k_\sigma\nabla\tilde{T}_\sigma\,\mathrm{d}A. \tag{35}$$

The forms of equations (34) and (35) that are valid in the *homogeneous* ω and η-regions are already available to us [25–32], and our objective at this point is to develop the forms that are valid everywhere in the system illustrated in Fig. 1.

The functional dependence of $\tilde{T}_\beta$ and $\tilde{T}_\sigma$ will be very complex in the boundary region, and in order to gain some insight into the nature of the functional dependence, we draw upon previous studies of the two-equation model [26–28, 30, 31] in which the spatial deviations are expressed as

$$\tilde{T}_\beta = \mathbf{b}_{\beta\beta}\cdot\nabla\langle T_\beta\rangle^\beta + \mathbf{b}_{\beta\sigma}\cdot\nabla\langle T_\sigma\rangle^\sigma - s_\beta(\langle T_\beta\rangle^\beta - \langle T_\sigma\rangle^\sigma) \tag{36a}$$

$$\tilde{T}_\sigma = \mathbf{b}_{\sigma\beta}\cdot\nabla\langle T_\beta\rangle^\beta + \mathbf{b}_{\sigma\sigma}\cdot\nabla\langle T_\sigma\rangle^\sigma + s_\sigma(\langle T_\sigma\rangle^\sigma - \langle T_\beta\rangle^\beta). \tag{36b}$$

It is important to keep in mind that these representations are only valid in the *homogeneous* ω-region. In order to understand how to construct the form of the *generalized* β-phase transport equation, we substitute equation (36a) into the interfacial flux term in equation (34) and note that *one of the terms* that will be produced is given by

$$\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot k_\beta\nabla\tilde{T}_\beta\,\mathrm{d}A = -\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot k_\beta\nabla[s_\beta(\langle T_\beta\rangle^\beta - \langle T_\sigma\rangle^\sigma)]\,\mathrm{d}A + \cdots \tag{37}$$

This, in turn, will lead to

$$\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot k_\beta \nabla \tilde{T}_\beta \,\mathrm{d}A$$
$$= -\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot k_\beta (\nabla s_\beta)(\langle T_\beta\rangle^\beta - \langle T_\sigma\rangle^\sigma)\,\mathrm{d}A$$
$$-\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot k_\beta s_\beta \nabla\langle T_\beta\rangle^\beta \,\mathrm{d}A$$
$$+\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot k_\beta s_\beta \nabla\langle T_\sigma\rangle^\sigma \,\mathrm{d}A + \cdots \tag{38}$$

which illustrates the presence of a classical heat exchange term that is proportional to $\langle T_\beta\rangle^\beta - \langle T_\sigma\rangle^\sigma$, along with *convective-like* terms that are proportional to $\nabla\langle T_\beta\rangle^\beta$ and $\nabla\langle T_\sigma\rangle^\sigma$. We can express the β-phase convective-like transport as

$$\frac{1}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot k_\beta s_\beta \nabla\langle T_\beta\rangle^\beta \,\mathrm{d}A$$
$$= (\rho c_\mathrm{p})_\beta \left[\frac{\alpha_\beta}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot s_\beta \nabla\langle T_\beta\rangle^\beta \,\mathrm{d}A\right] \tag{39}$$

in which α_β is the thermal diffusivity for the β-phase. We now define a 'velocity' according to

$$(\rho c_\mathrm{p})_\beta \left[\frac{\alpha_\beta}{\mathscr{V}}\int_{A_{\beta\sigma}} \mathbf{n}_{\beta\sigma}\cdot s_\beta \nabla\langle T_\beta\rangle^\beta \,\mathrm{d}A\right]$$
$$= (\rho c_\mathrm{p})_\beta \nabla\cdot(\mathbf{u}'_{\beta\beta}\langle T_\beta\rangle^\beta). \tag{40}$$

Here $\mathbf{u}'_{\beta\beta}$ is only *one* of the velocities that will appear in equation (34) on the basis of the representation given by equation (36a). It should be clear that $\mathbf{u}'_{\beta\beta}$ will undergo significant variations in the boundary region; however, in the homogeneous ω-region this velocity will be essentially constant. If one explores *all the possibilities* associated with the representation for $\tilde{T}_\beta$, one is motivated to express the generalized thermal energy transport equation for the β-phase as

Generalized β-phase transport equation

$$\varepsilon_\beta(\rho c_\mathrm{p})_\beta \frac{\partial\langle T_\beta\rangle^\beta}{\partial t} + (\rho c_\mathrm{p})_\beta \nabla\cdot(\langle \mathbf{v}_\beta\rangle\langle T_\beta\rangle^\beta)$$
$$-(\rho c_\mathrm{p})_\beta \nabla\cdot(\mathbf{u}_{\beta\beta}\langle T_\beta\rangle^\beta) - (\rho c_\mathrm{p})_\beta \nabla\cdot(\mathbf{u}_{\beta\sigma}\langle T_\sigma\rangle^\sigma)$$
$$= \nabla\cdot(\mathbf{K}^*_{\beta\beta}\cdot\nabla\langle T_\beta\rangle^\beta + \mathbf{K}^*_{\beta\sigma}\cdot\nabla\langle T_\sigma\rangle^\sigma)$$
$$-a_\mathrm{v}h(\langle T_\beta\rangle^\beta - \langle T_\sigma\rangle^\sigma). \tag{41}$$

In dealing with the conductive and dispersive transport, we have followed the nomenclature used by Quintard and Whitaker [30] with the exception that we have added an asterisk in order to indicate that these terms represent both *conduction and convection.* This is consistent with previous studies [29], and it is needed to clearly distinguish the β-phase transport process from the σ-phase transport process that does not contain any convective or dispersive transport. The nomenclature used for the 'velocity-like' terms differs slightly from that employed by Quintard *et al.* [31] and the correspondence is given by

this work	Quintard *et al.*
$(\rho c_\mathrm{p})_\beta \mathbf{u}_{\beta\beta}$	$\mathbf{u}_{\beta\beta}$
$(\rho c_\mathrm{p})_\beta \mathbf{u}_{\beta\sigma}$	$\mathbf{u}_{\beta\sigma}$

One can repeat the line of thought leading to equation (41) in order to develop the analogous transport equation for the σ-phase.

Generalized σ-phase transport equation

$$\varepsilon_\sigma(\rho c_\mathrm{p})_\sigma \frac{\partial\langle T_\sigma\rangle^\sigma}{\partial t} - (\rho c_\mathrm{p})_\sigma \nabla\cdot(\mathbf{u}_{\sigma\beta}\langle T_\beta\rangle^\beta)$$
$$-(\rho c_\mathrm{p})_\sigma \nabla\cdot(\mathbf{u}_{\sigma\sigma}\langle T_\sigma\rangle^\sigma) = \nabla\cdot(\mathbf{K}_{\sigma\beta}\cdot\nabla\langle T_\beta\rangle^\beta$$
$$+\mathbf{K}_{\sigma\sigma}\cdot\nabla\langle T_\sigma\rangle^\sigma) - a_\mathrm{v}h(\langle T_\sigma\rangle^\sigma - \langle T_\beta\rangle^\beta). \tag{42}$$

The correspondence between the velocities used in this expression and those used by Quintard *et al.* [31] is given by

this work	Quintard *et al.*
$(\rho c_\mathrm{p})_\sigma \mathbf{u}_{\sigma\beta}$	$\mathbf{u}_{\sigma\beta}$
$(\rho c_\mathrm{p})_\sigma \mathbf{u}_{\sigma\sigma}$	$\mathbf{u}_{\sigma\sigma}$

It is important to keep in mind that the coefficients that appear in equations (41) and (42) will undergo rapid variations in the *boundary region*, and their functional dependence in that region is not well understood. For example, in the *boundary region* the dispersion tensor, $\mathbf{K}^*_{\beta\beta}$, may depend on $\nabla\langle T_\beta\rangle^\beta$, and the velocity coefficient, $\mathbf{u}_{\beta\sigma}$, may depend on $\langle T_\beta\rangle^\beta - \langle T_\sigma\rangle^\sigma$, while in the *homogeneous* ω and η-regions these coefficients will be well behaved and predictable (either by theory or experiment). *Outside of the boundary region*, the non-local dispersion is zero, while *inside of the boundary region* the non-local dispersion is distributed among the terms in equation (41) in an unknown manner and the coefficients can only be determined by laboratory or numerical experiments. It is important to keep in mind that the derivation of equations (34) and (35) is quite rigorous and one should think of equations (41) and (42) as correct *by definition*, i.e. the values and functional dependence of the coefficients in those equations are such that the volume average temperatures predicted by equations (41) and (42) are identical to the true values. This is an acceptable point of view in terms of the theoretical analysis; however, jump conditions generally require experimental measurements to complete the closure. If the functional dependence of the coefficients is unclear or exceedingly complex, the experimental part of this problem will be difficult to accomplish.

2. DEVELOPMENT OF THE FLUX JUMP CONDITIONS

Given the generalized transport equations for the β and σ-phases, we are ready to develop the jump conditions. The coefficients in equations (41) and (42) undergo extremely rapid variations in the *boundary*

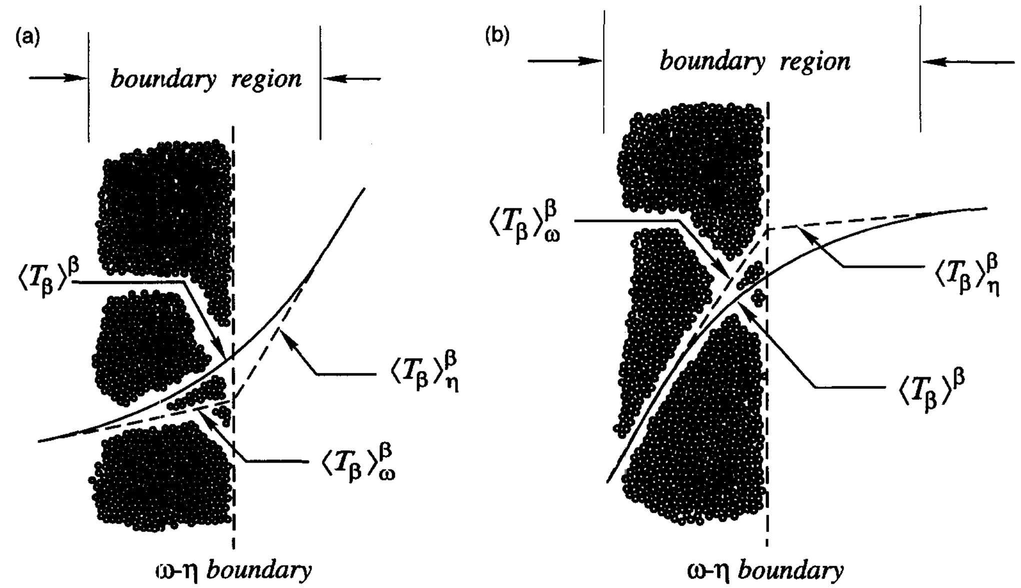

Fig. 4. Temperature profiles in the boundary region (a) $\mathbf{K}^*_{\beta\beta}+\mathbf{K}^*_{\beta\sigma} > k_\beta$; (b) $\mathbf{K}^*_{\beta\beta}+\mathbf{K}^*_{\beta\sigma} < k_\beta$.

region, in the same way that the stress undergoes extremely rapid variations in the *neighborhood of a phase interface* [38]. To avoid the difficulty associated with these rapid variations, we will apply the transport equations that are valid in the homogeneous parts of the ω and η-regions to the *entire space* occupied by the ω and η-regions. For example, this means that the computed values of $\langle T_\beta\rangle^\beta_\omega$ and $\langle T_\beta\rangle^\beta_\eta$ in the boundary region will not be equal to the value of $\langle T_\beta\rangle^\beta$ that *would be determined* by equation (41). We have indicated this situation in Fig. 4 where temperature profiles are illustrated for two cases, $\mathsf{K}^*_{\beta\beta}+\mathsf{K}^*_{\beta\sigma} > k_\beta$ and $\mathsf{K}^*_{\beta\beta}+\mathsf{K}^*_{\beta\sigma} < k_\beta$. It is important to understand that the profiles for $\langle T_\beta\rangle^\beta_\omega$ and $\langle T_\beta\rangle^\beta_\eta$ *are not extrapolations*, but are solutions to the transport equations that are valid in the homogeneous ω and η-regions and *applied everywhere*. Since the true volume average temperature will undoubtedly be a continuous function of position [29, 39], we will require that $\langle T_\beta\rangle^\beta_\omega$ and $\langle T_\beta\rangle^\beta_\eta$ form a continuous profile as indicated in Fig 4, and we will construct a flux jump condition at the ω-η boundary which requires that equations (41) and (42) be satisfied *on the average*.

We begin by listing the equations that are valid in the homogeneous ω and η-regions and the boundary conditions for the temperature and velocity.

ω-region
β-phase transport equation

$$\varepsilon_{\beta\omega}(\rho c_p)_\beta\frac{\partial\langle T_\beta\rangle^\beta_\omega}{\partial t}+(\rho c_p)_\beta\nabla\cdot(\langle\mathbf{v}_\beta\rangle_\omega\langle T_\beta\rangle^\beta_\omega)$$
$$-(\rho c_p)_\beta\nabla\cdot(\mathbf{u}_{\beta\beta\omega}\langle T_\beta\rangle^\beta_\omega)-(\rho c_p)_\beta\nabla\cdot(\mathbf{u}_{\beta\sigma\omega}\langle T_\sigma\rangle^\sigma_\omega)$$
$$=\nabla\cdot(\mathbf{K}^*_{\beta\beta\omega}\cdot\nabla\langle T_\beta\rangle^\beta_\omega+\mathbf{K}^*_{\beta\sigma\omega}\cdot\nabla\langle T_\sigma\rangle^\sigma_\omega)$$
$$-(a_v h)_\omega(\langle T_\beta\rangle^\beta_\omega-\langle T_\sigma\rangle^\sigma_\omega) \quad (43)$$

σ-phase transport equation

$$\varepsilon_{\sigma\omega}(\rho c_p)_\sigma\frac{\partial\langle T_\sigma\rangle^\sigma_\omega}{\partial t}-(\rho c_p)_\sigma\nabla\cdot(\mathbf{u}_{\sigma\beta\omega}\langle T_\beta\rangle^\beta_\omega)$$
$$-(\rho c_p)_\sigma\nabla\cdot(\mathbf{u}_{\sigma\sigma\omega}\langle T_\sigma\rangle^\sigma_\omega)=\nabla\cdot(\mathbf{K}_{\sigma\beta\omega}\cdot\nabla\langle T_\beta\rangle^\beta_\omega$$
$$+\mathbf{K}_{\sigma\sigma\omega}\cdot\nabla\langle T_\sigma\rangle^\sigma_\omega)-(a_v h)_\omega(\langle T_\sigma\rangle^\sigma_\omega-\langle T_\beta\rangle^\beta_\omega) \quad (44)$$

boundary conditions

B.C.1 $\langle T_\beta\rangle^\beta_\omega=\langle T_\beta\rangle^\beta_\eta$ at the ω–η boundary (45)

B.C.2 no condition on $\langle T_\sigma\rangle^\sigma_\omega$ at the ω–η boundary (46)

B.C.3 $\langle\mathbf{v}_\beta\rangle_\omega=\langle\mathbf{v}_\beta\rangle_\eta$, at the ω–η boundary (47)

η-region
β-phase transport equation

$$(\rho c_p)_\beta\frac{\partial\langle T_\beta\rangle^\beta_\eta}{\partial t}+(\rho c_p)_\beta\nabla\cdot(\langle\mathbf{v}_\beta\rangle_\eta\langle T_\beta\rangle^\beta_\eta)$$
$$=\nabla\cdot(k_\beta\nabla\langle T_\beta\rangle^\beta_\eta) \quad (48)$$

σ-phase transport equation
no homogeneous σ-phase transport equation in the η-region.

The length-scale constraints that must be satisfied in order that equations (43) and (44) are valid in the homogeneous ω-region are documented elsewhere [25–32]; however, the length-scale constraints associated with equation (48) are not so well known and are discussed in the following paragraphs.

2.1. *Homogeneous η-region*

It is important to note that the energy equation in the homogeneous η-region has exactly the same form as the original point equation given by equation (1). This is based on the approximation that the local volume averaged values in the homogeneous η-region are equal to the corresponding point values, i.e.

$$\langle \psi_\beta \rangle^\beta|_{\mathbf{x}} = \langle \psi_\beta \rangle|_{\mathbf{x}} = \psi_\beta|_{\mathbf{x}} \quad \text{in the homogeneous } \eta\text{-region.} \tag{49}$$

The justification of this result is given by Ochoa-Tapia and Whitaker [1, 2] who showed that average and point values in the homogeneous η-region are related by

$$\langle \psi_\beta \rangle|_{\mathbf{x}} = \psi_\beta|_{\mathbf{x}} + \tfrac{1}{2}\langle \mathbf{y}_\beta \mathbf{y}_\beta \rangle : \nabla\nabla\psi_\beta|_{\mathbf{x}} + \cdots \tag{50}$$

and this means that equation (49) is valid whenever

$$\langle \mathbf{y}_\beta \mathbf{y}_\beta \rangle : \nabla\nabla\psi_\beta \ll \psi_\beta \tag{51}$$

For example, if the temperature in the homogeneous η-region is a *linear function of position*, the restriction given by equation (51) is automatically satisfied and we conclude from equation (49) that

$$\langle T_\beta \rangle^\beta = \langle T_\beta \rangle = T_\beta, \quad \text{in the homogeneous } \eta\text{-region.} \tag{52}$$

Here it is understood that the average and the point temperatures are evaluated at the same position.

In the process of extracting equation (48) from equation (41), we have made use of

$$\langle \mathbf{v}_\beta T_\beta \rangle = \mathbf{v}_\beta T_\beta \quad \text{in the homogeneous } \eta\text{-region} \tag{53}$$

in order to conclude that there is no dispersion in the homogeneous η-region. A little thought will indicate that $\langle \mathbf{y}_\beta \mathbf{y}_\beta \rangle = \mathbf{O}(r_0^2)$ for a spherical averaging volume, and this means that the constraint given by equation (51) takes the form

$$r_0^2 \nabla\nabla\psi_\beta \ll \psi_\beta. \tag{54}$$

We can make use of the estimates given by Ochoa-Tapia and Whitaker [1, 2]

$$\nabla\psi_\beta = \mathbf{O}\left(\frac{\Delta\psi_\beta}{L_\psi}\right),$$

$$\nabla(\nabla\psi_\beta) = \mathbf{O}\left(\frac{\Delta(\nabla\psi_\beta)}{L_{\psi 1}}\right) = \mathbf{O}\left(\frac{\Delta\psi_\beta}{L_{\psi 1} L_\psi}\right) \tag{55}$$

along with equation (54) to conclude that the condition indicated by equation (49) is valid whenever the following length-scale constraint is satisfied.

$$\frac{r_0^2}{L_{\psi 1} L_\psi} \frac{\Delta\psi_\beta}{\psi_\beta} \ll 1, \quad \text{in the homgeneous } \eta\text{-region} \tag{56}$$

Here we note that when ψ_β is a *linear function of position*, the characteristic distance $L_{\psi 1}$ must be set equal to infinity in order to be consistent with the convention used in equation (55). If ψ_β is a *non-linear function of position*, and the characteristic lengths associated with ψ_β are *not large* compared to r_0, the constraint given by equation (56) may be difficult to satisfy. Before moving on to the jump condition based on equations (41) through (48), it is important to remark that the length-scale constraints associated with ψ_β are applied *only* in the homogeneous η-region and that no length-scale constraints have been imposed on the generalized energy transport equations *in the boundary region*.

2.2. *Jump condition*

The development of the jump conditions for the β and σ-phase transport equations is quite complicated and is given in the appendix. The procedure follows that used in the derivation of interfacial jump conditions [38, 40], and the objective is to obtain conditions that will require that equations (41) and (42) are satisfied *on the average* in the boundary region. In the appendix we show that the jump condition for the β-phase is given by:

Jump condition for the β-phase

$$\begin{aligned}
&\underbrace{(\rho c_\mathrm{p})_{\beta\mathrm{s}} \frac{\partial \langle T_\beta \rangle_\mathrm{s}^\beta}{\partial t}}_{\text{excess surface accumulation}} \\
&+ \underbrace{\nabla_\mathrm{s} \cdot [(\rho c_\mathrm{p})_{\beta\mathrm{s}} \langle \mathbf{v}_\beta \rangle_\mathrm{s} \langle T_\beta \rangle_\mathrm{s}^\beta - (\mathbf{K}^*_{\beta\beta\mathrm{s}} \cdot \nabla_\mathrm{s} \langle T_\beta \rangle_\mathrm{s}^\beta + \mathbf{K}^*_{\beta\sigma\mathrm{s}} \cdot \nabla_\mathrm{s} \langle T_\sigma \rangle_\mathrm{s}^\sigma)]}_{\text{excess surface convective and conductive transport}} \\
&- \underbrace{\nabla_\mathrm{s} \cdot [(\rho c_\mathrm{p})_{\beta\mathrm{s}} (\mathbf{u}_{\beta\beta})_\mathrm{s} \langle T_\beta \rangle_\mathrm{s}^\beta + (\rho c_\mathrm{p})_{\beta\mathrm{s}} (\mathbf{u}_{\beta\sigma})_\mathrm{s} \langle T_\sigma \rangle_\mathrm{s}^\sigma]}_{\text{excess surface convective-like transport}} \\
&+ \underbrace{\mathbf{n}_{\omega\eta} \cdot ((\rho c_\mathrm{p})_\beta \mathbf{u}_{\beta\beta\omega} \langle T_\beta \rangle_\omega^\beta + (\rho c_\mathrm{p})_\beta \mathbf{u}_{\beta\sigma\omega} \langle T_\sigma \rangle_\omega^\sigma)}_{\text{normal flux of convective-like transport}} \\
&+ \mathbf{n}_{\omega\eta} \cdot (\mathbf{K}^*_{\beta\beta\omega} \cdot \nabla \langle T_\beta \rangle_\omega^\beta + \mathbf{K}^*_{\beta\sigma\omega} \cdot \nabla \langle T_\sigma \rangle_\omega^\sigma) \\
&= \mathbf{n}_{\omega\eta} \cdot (k_\beta \nabla \langle T_\beta \rangle_\eta^\beta) - \underbrace{(a_\mathrm{v} h)_\mathrm{s} (\langle T_\beta \rangle_\mathrm{s}^\beta - \langle T_\sigma \rangle_\mathrm{s}^\sigma)}_{\text{excess surface heat exchange}}
\end{aligned} \tag{57}$$

while that for the σ-phase takes the form:

Jump condition for the σ-phase

$$\begin{aligned}
&\underbrace{(\rho c_\mathrm{p})_{\sigma\mathrm{s}} \frac{\partial \langle T_\sigma \rangle_\mathrm{s}^\sigma}{\partial t}}_{\text{excess surface accumulation}} \\
&- \underbrace{\nabla_\mathrm{s} \cdot [(\mathbf{K}_{\sigma\beta\mathrm{s}} \cdot \nabla_\mathrm{s} \langle T_\beta \rangle_\mathrm{s}^\beta + \mathbf{K}_{\sigma\sigma\mathrm{s}} \cdot \nabla_\mathrm{s} \langle T_\sigma \rangle_\mathrm{s}^\sigma)]}_{\text{excess surface conductive transport}} \\
&- \underbrace{\nabla_\mathrm{s} \cdot [(\rho c_\mathrm{p})_{\sigma\mathrm{s}} (\mathbf{u}_{\sigma\beta})_\mathrm{s} \langle T_\beta \rangle_\mathrm{s}^\beta + (\rho c_\mathrm{p})_{\sigma\mathrm{s}} (\mathbf{u}_{\sigma\sigma})_\mathrm{s} \langle T_\sigma \rangle_\mathrm{s}^\sigma]}_{\text{excess surface convective-like transport}}
\end{aligned}$$

$$+\underbrace{\mathbf{n}_{\omega\eta}\cdot[(\rho c_{\mathrm{p}})_\sigma \mathbf{u}_{\sigma\beta\omega}\langle T_\beta\rangle_\omega^\beta+(\rho c_{\mathrm{p}})_\sigma \mathbf{u}_{\sigma\sigma\omega}\langle T_\sigma\rangle_\omega^\sigma]}_{\text{normal flux of convective-like transport}}$$

$$+\mathbf{n}_{\omega\eta}\cdot(\mathbf{K}_{\sigma\beta\omega}\cdot\nabla\langle T_\beta\rangle_\omega^\beta+\mathbf{K}_{\sigma\sigma\omega}\cdot\nabla\langle T_\sigma\rangle_\omega^\sigma)$$

$$=\underbrace{(a_{\mathrm{v}}h)_{\mathrm{s}}(\langle T_\beta\rangle_{\mathrm{s}}^\beta-\langle T_\sigma\rangle_{\mathrm{s}}^\sigma)}_{\text{excess surface heat exchange}} \quad (58)$$

A key term in both these jump conditions is the *excess surface heat exchange*. The coefficient, $(a_{\mathrm{v}}h)_{\mathrm{s}}$, has the units of a traditional heat transfer coefficient, and it is a measure of the rate of heat exchange between the β-phase and the σ-phase *in the neighborhood of* the ω–η boundary. We will identify this coefficient as

$$(a_{\mathrm{v}}h)_{\mathrm{s}}=h_{\beta\sigma} \quad (59)$$

with the idea that it represents the heat exchange between the β and σ-phases in the boundary region.

Purely on the basis of intuition, we will retain the *excess surface heat exchange* in both equation (57) and in equation (58), but we will *discard* all the remaining excess surface terms along with the normal flux of the convective-like terms. Under these circumstances our two jump conditions can be expressed in the form

$$\mathbf{n}_{\omega\eta}\cdot(\mathbf{K}^*_{\beta\beta\omega}\cdot\nabla\langle T_\beta\rangle_\omega^\beta+\mathbf{K}^*_{\beta\sigma\omega}\cdot\nabla\langle T_\sigma\rangle_\omega^\sigma)$$
$$=\mathbf{n}_{\omega\eta}\cdot(k_\beta\nabla\langle T_\beta\rangle_\eta^\beta)-h_{\beta\sigma}(\langle T_\beta\rangle_\omega^\beta-\langle T_\sigma\rangle_\omega^\sigma)$$
$$\text{at the } \omega\text{–}\eta \text{ boundary} \quad (60)$$

$$\mathbf{n}_{\omega\eta}\cdot(\mathbf{K}_{\sigma\beta\omega}\cdot\nabla\langle T_\beta\rangle_\omega^\beta+\mathbf{K}_{\sigma\sigma\omega}\cdot\nabla\langle T_\sigma\rangle)$$
$$=h_{\beta\sigma}(\langle T_\beta\rangle_\omega^\beta-\langle T_\sigma\rangle_\omega^\sigma) \quad \text{at the } \omega\text{–}\eta \text{ boundary.} \quad (61)$$

Here we have used the fact that the volume averaged temperatures are continuous, thus the surface temperatures are specified according to

$$\langle T_\beta\rangle_{\mathrm{s}}^\beta=\langle T_\beta\rangle_\omega^\beta=\langle T_\beta\rangle_\eta^\beta \quad \text{at the } \omega\text{–}\eta \text{ boundary} \quad (62)$$

$$\langle T_\sigma\rangle_{\mathrm{s}}^\sigma=\langle T_\sigma\rangle_\omega^\sigma \quad \text{at the } \omega\text{–}\eta \text{ boundary.} \quad (63)$$

From equations (60) and (61) we see that the excess surface heat exchange term controls how the heat flux between the ω and η-regions is distributed between the β and σ-phases. This means that a complete understanding of the heat flux boundary conditions associated with the two-equation model requires the knowledge of the boundary heat transfer coefficient, $h_{\beta\sigma}$. Considerable information is available concerning the term $a_{\mathrm{v}}h$ and therefore the heat transfer coefficient, h [25, 28, 31, 41–46]; however, it would appear that very little is known concerning $h_{\beta\sigma}$. It seems likely that h and $h_{\beta\sigma}$ are the same order of magnitude, but detailed studies of the flux jump conditions given by equations (60) and (61) are required in order to develop a complete understanding of the boundary heat transfer coefficient, $h_{\beta\sigma}$.

2.3. *Local gradient equilibrium*

Quintard and Whitaker [37] have suggested that the approximation indicated by

$$\nabla\langle T_\beta\rangle_\omega^\beta=\nabla\langle T_\sigma\rangle_\omega^\sigma \quad (64)$$

may be valid even when the condition of local thermal equilibrium is not satisfied, i.e. $\langle T_\beta\rangle_\omega^\beta\neq\langle T_\sigma\rangle_\omega^\sigma$. The *restrictions* [47] associated with the *assumption* given by equation (64) are easy to identify; however, the *constraints* associated with local gradient equilibrium still need to be developed and verified. Even though the constraints are not yet available, equation (64) represents a popular simplification and when used with equations (43) through (48) and equations (60) and (61) it leads to the following form of our heat transfer problem

ω-region

β-phase transport equation

$$\varepsilon_{\beta\omega}(\rho c_{\mathrm{p}})_\beta\frac{\partial\langle T_\beta\rangle_\omega^\beta}{\partial t}+(\rho c_{\mathrm{p}})_\beta\nabla\cdot(\langle \mathbf{v}_\beta\rangle_\omega\langle T_\beta\rangle_\omega^\beta)$$
$$-(\rho c_{\mathrm{p}})_\beta\nabla\cdot[(\mathbf{u}_{\beta\beta\omega}+\mathbf{u}_{\beta\sigma\omega})\langle T_\beta\rangle_\omega^\beta]$$
$$=\nabla\cdot(\mathbf{K}^*_{\beta\omega}\cdot\nabla\langle T_\beta\rangle_\omega^\beta)-(a_{\mathrm{v}}h)_\omega(\langle T_\beta\rangle_\omega^\beta-\langle T_\sigma\rangle_\omega^\sigma) \quad (65a)$$

σ-phase transport equation

$$\varepsilon_{\sigma\omega}(\rho c_{\mathrm{p}})_\sigma\frac{\partial\langle T_\sigma\rangle_\omega^\sigma}{\partial t}-(\rho c_{\mathrm{p}})_\sigma\nabla\cdot[(\mathbf{u}_{\sigma\sigma\omega}+\mathbf{u}_{\sigma\beta\omega})\langle T_\sigma\rangle_\omega^\sigma]=$$
$$\nabla\cdot(\mathbf{K}_{\sigma\omega}\cdot\nabla\langle T_\sigma\rangle_\omega^\sigma)-(a_{\mathrm{v}}h)_\omega(\langle T_\sigma\rangle_\omega^\sigma-\langle T_\beta\rangle_\omega^\beta) \quad (65b)$$

conditions at the ω–η boundary

$$\text{B.C.1} \quad \langle T_\beta\rangle_\omega^\beta=\langle T_\beta\rangle_\eta^\beta \quad (65c)$$

$$\text{B.C.2} \quad \mathbf{n}_{\omega\eta}\cdot(\mathbf{K}^*_{\beta\omega}\cdot\nabla\langle T_\beta\rangle_\omega^\beta)$$
$$=\mathbf{n}_{\omega\eta}\cdot(k_\beta\nabla\langle T_\beta\rangle_\eta^\beta)-h_{\beta\sigma}(\langle T_\beta\rangle_\omega^\beta-\langle T_\sigma\rangle_\omega^\sigma) \quad (65d)$$

$$\text{B.C.3} \quad \mathbf{n}_{\omega\eta}\cdot(\mathbf{K}_{\sigma\omega}\cdot\nabla\langle T_\sigma\rangle_\omega^\sigma)=h_{\beta\sigma}(\langle T_\beta\rangle_\omega^\beta-\langle T_\sigma\rangle_\omega^\sigma) \quad (65e)$$

$$\text{B.C.4} \quad \langle \mathbf{v}_\beta\rangle_\omega=\langle \mathbf{v}_\beta\rangle_\eta \quad \text{at the } \omega\text{–}\eta \text{ boundary.} \quad (65f)$$

η-region

β-phase transport equation

$$(\rho c_{\mathrm{p}})_\beta\frac{\partial\langle T_\beta\rangle_\eta^\beta}{\partial t}+(\rho c_{\mathrm{p}})_\beta\nabla\cdot(\langle \mathbf{v}_\beta\rangle_\eta\langle T_\beta\rangle_\eta^\beta)$$
$$=\nabla\cdot(k_\beta\nabla\langle T_\beta\rangle_\eta^\beta) \quad (65g)$$

Here we have simplified the nomenclature by use of the following representations

$$\mathbf{K}^*_{\beta\omega}=\mathbf{K}^*_{\beta\beta\omega}+\mathbf{K}^*_{\beta\sigma\omega} \quad (66a)$$

$$\mathbf{K}_{\sigma\omega} = \mathbf{K}_{\sigma\beta\omega} + \mathbf{K}_{\sigma\sigma\omega}. \tag{66b}$$

2.4. *Local thermal equilibrium*

Local thermal equilibrium is characterized by the approximation

$$\langle T_\beta \rangle_\omega^\beta = \langle T_\sigma \rangle_\omega^\sigma = \langle T \rangle \tag{67}$$

and the constraints associated with this condition have been considered in depth [35–37]. When the condition of local thermal equilibrium is imposed on equations (65) we obtain

ω-region

$$[\varepsilon_{\beta\omega}(\rho c_p)_\beta + \varepsilon_{\beta\sigma}(\rho c_p)_\sigma] \frac{\partial \langle T \rangle}{\partial t} + (\rho c_p)_\beta \nabla \cdot (\langle \mathbf{v}_\beta \rangle_\omega \langle T \rangle) = \nabla \cdot (\mathbf{K}_\omega^* \cdot \nabla \langle T \rangle) \tag{68a}$$

boundary conditions

$$\text{B.C.1} \quad \langle T \rangle = \langle T_\beta \rangle_\eta^\beta \quad \text{at the } \omega\text{–}\eta \text{ boundary} \tag{68b}$$

$$\text{B.C.2} \quad \mathbf{n}_{\omega\eta} \cdot (\mathbf{K}_\omega^* \cdot \nabla \langle T \rangle) = \mathbf{n}_{\omega\eta} \cdot (k_\beta \nabla \langle T_\beta \rangle_\eta^\beta) \quad \text{at the } \omega\text{–}\eta \text{ boundary} \tag{68c}$$

$$\text{B.C.3} \quad \langle \mathbf{v}_\beta \rangle_\omega = \langle \mathbf{v}_\beta \rangle_\eta \quad \text{at the } \omega\text{–}\eta \text{ boundary.} \tag{68d}$$

η-region

β-phase transport equation

$$(\rho c_p)_\beta \frac{\partial \langle T_\beta \rangle_\eta^\beta}{\partial t} + (\rho c_p)_\beta \nabla \cdot (\langle \mathbf{v}_\beta \rangle_\eta \langle T_\beta \rangle_\eta^\beta) = \nabla \cdot (k_\beta \nabla \langle T_\beta \rangle_\eta^\beta). \tag{68e}$$

In this representation of the heat transfer problem we have combined the thermal dispersion tensors according to

$$\mathbf{K}_\omega^* = \mathbf{K}_{\beta\beta\omega}^* + \mathbf{K}_{\beta\sigma\omega}^* + \mathbf{K}_{\sigma\beta\omega} + \mathbf{K}_{\sigma\sigma\omega} \tag{69}$$

and we have discarded the convective-like terms on the basis of the work of Quintard and Whitaker [30].

3. CONCLUSIONS

In this work we have developed the flux jump conditions between a porous medium and a homogeneous fluid when the condition of local thermal equilibrium is not valid. Under these circumstances, separate transport equations are required for each phase. The jump conditions at the boundary between the porous medium and the homogeneous fluid contain an excess surface heat exchange term that controls the way in which the total flux is distributed between the two phases. Either careful experimental studies or detailed numerical experiments are needed to determine the excess surface heat transfer coefficient. As an estimate, $h_{\beta\sigma}$ can be equated to the traditional heat transfer coefficient for flow in porous media, and values of h are available from a variety of sources [25, 31, 37, 41, 42, 45, 46].

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APPENDIX

We begin the development of the energy flux condition, or the surface energy transport equation, with the generalized volume averaged energy equation for the β-phase written in the form

$$\frac{\partial}{\partial t}(\varepsilon_\beta(\rho c_p)_\beta\langle T_\beta\rangle^\beta)+\nabla\cdot(\varepsilon_\beta(\rho c_p)_\beta\langle \mathbf{v}_\beta\rangle^\beta\langle T_\beta\rangle^\beta)-\nabla\cdot((\rho c_p)_\beta\mathbf{u}_{\beta\beta}\langle T_\beta\rangle^\beta)-\nabla\cdot((\rho c_p)_\beta\mathbf{u}_{\beta\sigma}\langle T_\sigma\rangle^\sigma)$$
$$=\nabla\cdot(\mathbf{K}^*_{\beta\beta}\cdot\nabla\langle T_\beta\rangle^\beta+\mathbf{K}^*_{\beta\sigma}\cdot\nabla\langle T_\sigma\rangle^\sigma)-a_v h(\langle T_\beta\rangle^\beta-\langle T_\sigma\rangle^\sigma) \quad \text{(A1)}$$

It is convenient to use an *integral statement* of this result that is comparable to that used in the development of jump conditions at phase interfaces [38, 40]. To this end, we let $\mathscr{V}_\infty(t)$ be a volume bounded by a surface $\mathscr{A}_\infty(t)$ which has a speed of displacement [38] given by $\langle \mathbf{v}_\beta\rangle^\beta\cdot\mathbf{n}$ where $\mathbf{n}$ is the outwardly directed unit normal vector associated with the surface $\mathscr{A}_\infty(t)$. This volume is illustrated in Fig. A1, and in terms of $\mathscr{V}_\infty(t)$ the integral statement of equation (A1) is given by

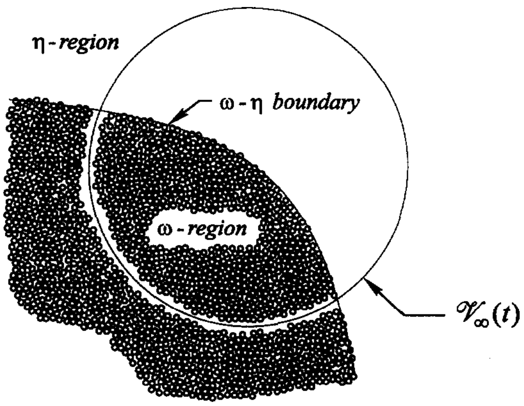

Fig. A1. Large-scale volume.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{V}_\infty(t)}(\varepsilon_\beta(\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta)\,\mathrm{d}V-\int_{\mathscr{A}_\infty(t)}\mathbf{n}((\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\beta}\langle T_\beta\rangle^\beta)\,\mathrm{d}A-\int_{\mathscr{A}_\infty(t)}\mathbf{n}\cdot((\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\sigma}\langle T_\sigma\rangle^\sigma)\,\mathrm{d}A$$
$$=\int_{\mathscr{A}_\infty(t)}\mathbf{n}\cdot(\mathbf{K}^*_{\beta\beta}\cdot\nabla\langle T_\beta\rangle^\beta+\mathbf{K}^*_{\beta\sigma}\cdot\nabla\langle T_\sigma\rangle^\sigma)\,\mathrm{d}A-\int_{\mathscr{V}_\infty(t)}a_\mathrm{v}h(\langle T_\beta\rangle^\beta-\langle T_\sigma\rangle^\sigma)\,\mathrm{d}V. \quad \text{(A2)}$$

The portion of $\mathscr{V}_\infty(t)$ that lies in the ω-region will be designated by $V_\omega(t)$ while the part that lies in the η-region will be identified by $V_\eta(t)$. It follows that

$$\mathscr{V}_\infty(t)=V_\omega(t)+V_\eta(t) \quad \text{(A3)}$$

The form of equation (A1) that is valid in the homogeneous ω-region is given by
ω-region

$$\frac{\partial}{\partial t}(\varepsilon_{\beta\omega}(\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta_\omega)+\nabla\cdot(\varepsilon_{\beta\omega}(\rho c_\mathrm{p})_\beta\langle \mathbf{v}_\beta\rangle^\beta_\omega\langle T_\beta\rangle^\beta_\omega)-\nabla\cdot((\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\beta\omega}\langle T_\beta\rangle^\beta_\omega)-\nabla\cdot((\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\sigma\omega}\langle T_\sigma\rangle^\sigma_\omega)$$
$$=\nabla\cdot(\mathbf{K}^*_{\beta\beta\omega}\cdot\nabla\langle T_\beta\rangle^\beta_\omega+\mathbf{K}^*_{\beta\sigma\omega}\cdot\nabla\langle T_\sigma\rangle^\sigma_\omega)-(a_\mathrm{v}h)_\omega(\langle T_\beta\rangle^\beta_\omega-\langle T_\sigma\rangle^\sigma_\omega) \quad \text{(A4)}$$

while for the homogeneous η-region we find
η-region

$$\frac{\partial}{\partial t}((\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta_\eta)+\nabla\cdot((\rho c_\mathrm{p})_\beta\langle \mathbf{v}_\beta\rangle^\beta_\eta\langle T_\beta\rangle^\beta_\eta)=\nabla\cdot(k_\beta\nabla\langle T_\beta\rangle^\beta_\eta). \quad \text{(A5)}$$

In order to develop the energy jump condition, we need to integrate equations (A4) and (A5) over $V_\omega(t)$ and $V_\eta(t)$, respectively, and then subtract those two integral equations from equation (A2). In this manner we will obtain a jump condition that can be used with equations (A4) and (A5) to produce a solution to the energy transport process that will satisfy the integral given by equation (A2), i.e. equation (A1) will be satisfied *on the average*. We can make use of the general transport theorem [48] to express the integral of equation (A4) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\omega(t)}(\varepsilon_{\beta\omega}(\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta_\omega)\,\mathrm{d}V+\int_{A_{\omega\eta}(t)}\mathbf{n}_{\omega\eta}\cdot(\varepsilon_{\beta\omega}(\rho c_\mathrm{p})_\beta\langle \mathbf{v}_\beta\rangle^\beta_\omega\langle T_\beta\rangle^\beta_\omega)\,\mathrm{d}A-\int_{A_\omega(t)}\mathbf{n}_\omega\cdot((\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\beta\omega}\langle T_\beta\rangle^\beta_\omega+(\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\sigma\omega}\langle T_\sigma\rangle^\sigma_\omega)\,\mathrm{d}A$$
$$-\int_{A_{\omega\eta}(t)}\mathbf{n}_{\omega\eta}\cdot((\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\beta\omega}\langle T_\beta\rangle^\beta_\omega+(\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\sigma\omega}\langle T_\sigma\rangle^\sigma_\omega)\,\mathrm{d}A$$
$$=\int_{A_\omega(t)}\mathbf{n}_\omega\cdot(\mathbf{K}^*_{\beta\beta\omega}\cdot\nabla\langle T_\beta\rangle^\beta_\omega+\mathbf{K}^*_{\beta\sigma\omega}\cdot\nabla\langle T_\sigma\rangle^\sigma_\omega)\,\mathrm{d}A+\int_{A_{\omega\eta}(t)}\mathbf{n}_{\omega\eta}\cdot(\mathbf{K}^*_{\beta\beta\omega}\cdot\nabla\langle T_\beta\rangle^\beta_\omega+\mathbf{K}^*_{\beta\sigma\omega}\cdot\nabla\langle T_\sigma\rangle^\delta_\omega)\,\mathrm{d}A$$
$$-\int_{V_\omega(t)}(a_\mathrm{v}h)_\omega(\langle T_\beta\rangle^\beta_\omega-\langle T_\sigma\rangle^\sigma_\omega)\,\mathrm{d}V \quad \text{(A6)}$$

while the integral of equation (A5) takes a somewhat simpler form given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\eta(t)}((\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta_\eta)\,\mathrm{d}V+\int_{A_{\eta\omega}(t)}\mathbf{n}_{\eta\omega}\cdot((\rho c_\mathrm{p})_\beta\langle \mathbf{n}_\beta\rangle^\beta_\eta\langle T_\beta\rangle^\beta_\eta)\,\mathrm{d}A=\int_{A_\eta(t)}\mathbf{n}_\eta\cdot(k_\beta\nabla\langle T_\beta\rangle^\beta_\eta)\,\mathrm{d}A+\int_{A_{\eta\omega}(t)}\mathbf{n}_{\eta\omega}\cdot(k_\beta\nabla\langle T_\beta\rangle^\beta_\eta)\,\mathrm{d}A. \quad \text{(A7)}$$

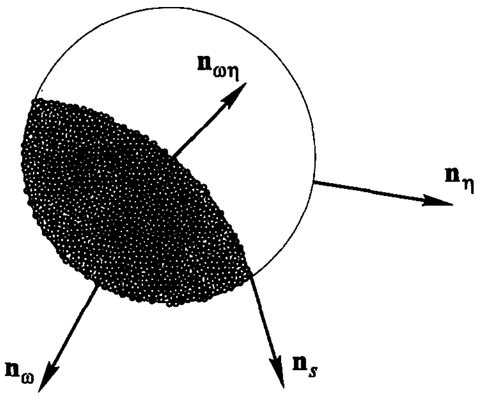

Fig. A2. Unit vectors.

The unit normal vectors contained in these results are identified in Fig. A2, and one should be careful to note that the *speed of displacement* of the surface, $A_{\omega\eta}(t) = A_{\eta\omega}(t)$, is zero for the system illustrated in Fig. A1.

Before subtracting equations (A6) and (A7) from equation (A2) in order to develop the jump condition, we should *define* the excess surface thermal energy according to

Excess surface thermal energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathscr{V}_\infty(t)} \varepsilon_\beta(\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta \,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\omega(t)} \varepsilon_{\beta\omega}(\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta_\omega \,\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\eta(t)} (\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta_\eta \,\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int_{A_{\omega\eta}(t)} (\rho c_\mathrm{p})_{\beta s}\langle T_\beta\rangle^\beta_s \,\mathrm{d}V. \quad (\mathrm{A8})$$

Here we note that the term on the left hand side represents *all the thermal energy* contained in the volume $\mathscr{V}_\infty(t)$. The first two terms on the right hand side represent the thermal energy contained in $\mathscr{V}_\infty(t)$ *as determined by* equations (A4) and (A5) and since these equations are not valid in the boundary region the first two terms on the right hand side of equation (A8) will not necessarily be equal to the term on the left hand side. Inclusion of the excess surface thermal energy in the jump condition will assure us that the total time rate of change of thermal energy will be given correctly by the solution of equations (A4) and (A5).

Equation (A8) is the preferred representation of an excess function; however, sometimes it is convenient to use the alternate form given by

Excess surface thermal energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\omega(t)} (\varepsilon_\beta(\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta - \varepsilon_{\beta\omega}(\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta_\omega) \,\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\eta(t)} (\varepsilon_\beta(\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta - (\rho c_\mathrm{p})_\beta\langle T_\beta\rangle^\beta_\eta) \,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{A_{\omega\eta}(t)} (\rho c_\mathrm{p})_{\beta s}\langle T_\beta\rangle^\beta_s \,\mathrm{d}V.$$

(A9)

When we subtract equations (A6) and (A7) from equation (A2) and make use of the definition of the excess surface thermal energy given by equation (A8), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{A_{\omega\eta}(t)} (\rho c_\mathrm{p})_{\beta s}\langle T_\beta\rangle^\beta_s \,\mathrm{d}V - \int_{A_{\omega\eta}(t)} \mathbf{n}_{\omega\eta}\cdot((\rho c_\mathrm{p})_\beta\langle \mathbf{v}_\beta\rangle_\omega\langle T_\beta\rangle^\beta_\omega - (\rho c_\mathrm{p})_\beta\langle \mathbf{v}_\beta\rangle_\eta\langle T_\beta\rangle^\beta_\eta)\,\mathrm{d}A \\
&\quad - \int_{A_\omega(t)} \mathbf{n}_\omega\cdot[((\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\beta}\langle T_\beta\rangle^\beta - (\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\beta\omega}\langle T_\beta\rangle^\beta_\omega) + ((\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\sigma}\langle T_\sigma\rangle^\sigma - (\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\sigma\omega}\langle T_\sigma\rangle^\sigma_\omega)]\,\mathrm{d}A \\
&\quad - \int_{A_\eta(t)} \mathbf{n}_\eta\cdot[((\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\beta}\langle T_\beta\rangle^\beta - 0) + ((\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\sigma}\langle T_\sigma\rangle^\sigma - 0)]\,\mathrm{d}A + \int_{A_{\omega\eta}(t)} \mathbf{n}_{\omega\eta}\cdot((\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\beta\omega}\langle T_\beta\rangle^\beta_\omega + (\rho c_\mathrm{p})_\beta\mathbf{u}_{\beta\sigma\omega}\langle T_\sigma\rangle^\sigma_\omega)\,\mathrm{d}A \\
&= \int_{A_\omega(t)} \mathbf{n}_\omega\cdot[(\mathbf{K}^*_{\beta\beta}\cdot\nabla\langle T_\beta\rangle^\beta + \mathbf{K}^*_{\beta\sigma}\cdot\nabla\langle T_\sigma\rangle^\sigma) - (\mathbf{K}^*_{\beta\beta\omega}\cdot\nabla\langle T_\beta\rangle^\beta_\omega + \mathbf{K}^*_{\beta\sigma\omega}\cdot\nabla\langle T_\sigma\rangle^\sigma_\omega)]\,\mathrm{d}A \\
&\quad + \int_{A_\eta(t)} \mathbf{n}_\eta\cdot[(\mathbf{K}^*_{\beta\beta}\cdot\nabla\langle T_\beta\rangle^\beta + \mathbf{K}^*_{\beta\sigma}\cdot\nabla\langle T_\sigma\rangle^\sigma) - k_\beta\nabla\langle T_\beta\rangle^\beta_\eta]\,\mathrm{d}A - \int_{A_{\omega\eta}(t)} \mathbf{n}_{\omega\eta}\cdot[(\mathbf{K}^*_{\beta\beta\omega}\cdot\nabla\langle T_\beta\rangle^\beta_\omega + \mathbf{K}^*_{\beta\sigma\omega}\langle T_\sigma\rangle^\sigma_\omega) - k_\beta\nabla\langle T_\beta\rangle^\beta_\eta]\,\mathrm{d}A \\
&\quad - \int_{V_\omega(t)} [a_\mathrm{v}h(\langle T_\beta\rangle^\beta - \langle T_\sigma\rangle^\sigma) - (a_\mathrm{v}h)_\omega(\langle T_\beta\rangle^\beta_\omega - \langle T_\sigma\rangle^\sigma_\omega)]\,\mathrm{d}V - \int_{V_\eta(t)} [a_\mathrm{v}h(\langle T_\beta\rangle^\beta - \langle T_\sigma\rangle^\sigma) - 0]\,\mathrm{d}V. \quad (\mathrm{A10})
\end{aligned}$$

Here we have used strategically placed zeros in order to clearly identify those groups of terms that have the form indicated by the left hand side of equation (A9) and can therefore be represented in terms of excess surface functions. We begin with the convective-like terms and *define* the surface excess transport as

Excess surface convective-like transport

$$\oint_C \mathbf{n}_s \cdot [(\rho c_p)_{\beta s}(\mathbf{u}_{\beta\beta})_s \langle T_\beta \rangle_s^\beta + (\rho c_p)_{\beta s}(\mathbf{u}_{\beta\sigma})_s \langle T_\sigma \rangle_s^\sigma]\, d\sigma$$

$$= \int_{A_\omega(t)} \mathbf{n}_\omega \cdot [((\rho c_p)_\beta \mathbf{u}_{\beta\beta} \langle T_\beta \rangle^\beta - (\rho c_p)_\beta \mathbf{u}_{\beta\beta\omega} \langle T_\beta \rangle_\omega^\beta) + ((\rho c_p)_\beta \mathbf{u}_{\beta\sigma} \langle T_\sigma \rangle^\sigma - (\rho c_p)_\beta \mathbf{u}_{\beta\sigma\omega} \langle T_\sigma \rangle_\omega^\sigma)]\, dA$$

$$+ \int_{A_\eta(t)} \mathbf{n}_\eta \cdot [((\rho c_p)_\beta \mathbf{u}_{\beta\beta} \langle T_\beta \rangle^\beta - 0) + ((\rho c_p)_\beta \mathbf{u}_{\beta\sigma} \langle T_\sigma \rangle^\sigma - 0)]\, dA. \quad \text{(A11)}$$

The conductive transport terms in equation (A10) naturally lead to the definition

Excess surface conductive transport

$$\oint_C \mathbf{n}_s \cdot (\mathbf{K}^*_{\beta\beta s} \cdot \nabla_s \langle T_\beta \rangle_s^\beta + \mathbf{K}^*_{\beta\sigma s} \cdot \nabla_s \langle T_\sigma \rangle_s^\sigma)\, d\sigma$$

$$= \int_{A_\omega(t)} \mathbf{n}_\omega \cdot [(\mathbf{K}^*_{\beta\beta} \cdot \nabla \langle T_\beta \rangle^\beta + \mathbf{K}^*_{\beta\sigma} \cdot \nabla \langle T_\sigma \rangle^\sigma) - (\mathbf{K}^*_{\beta\beta\omega} \cdot \nabla \langle T_\beta \rangle_\omega^\beta + \mathbf{K}^*_{\beta\sigma\omega} \cdot \nabla \langle T_\sigma \rangle_\omega^\sigma)]\, dA$$

$$+ \int_{A_\eta(t)} \mathbf{n}_\eta \cdot [(\mathbf{K}^*_{\beta\beta} \cdot \nabla \langle T_\beta \rangle^\beta + \mathbf{K}^*_{\beta\sigma} \cdot \nabla \langle T_\sigma \rangle^\sigma) - (k_\beta \nabla \langle T_\beta \rangle_\eta^\beta)]\, dA. \quad \text{(A12)}$$

Finally we define the excess surface heat exchange according to

Excess surface heat exchange

$$\int_{A_{\omega\eta}(t)} (a_v h)_s (\langle T_\beta \rangle_s^\beta - \langle T_\sigma \rangle_s^\sigma)\, dA = \int_{V_\omega(t)} \{[(a_v h)(\langle T_\beta \rangle^\beta - \langle T_\sigma \rangle^\sigma)] - [(a_v h)_\omega (\langle T_\beta \rangle_\omega^\beta - \langle T_\sigma \rangle_\omega^\sigma)]\}\, dV$$

$$+ \int_{V_\eta(t)} \{[(a_v h)(\langle T_\beta \rangle^\beta - \langle T_\sigma \rangle^\sigma)] - [0]\}\, dV. \quad \text{(A13)}$$

With these definitions the integral form of the jump condition can be expressed as

$$\frac{d}{dt} \int_{A_{\omega\eta}(t)} (\rho c_p)_{\beta s} \langle T_\beta \rangle_s^\beta\, dV - \int_{A_{\omega\eta}(t)} \mathbf{n}_{\omega\eta} \cdot ((\rho c_p)_\beta \langle \mathbf{v}_\beta \rangle_\omega \langle T_\beta \rangle_\omega^\beta - (\rho c_p)_\beta \langle \mathbf{v}_\beta \rangle_\eta \langle T_\beta \rangle_\eta^\beta)\, dA$$

$$- \oint_C \mathbf{n}_s \cdot [(\rho c_p)_{\beta s}(\mathbf{u}_{\beta\beta})_s \langle T_\beta \rangle_s^\beta + (\rho c_p)_{\beta s}(\mathbf{u}_{\beta\sigma})_s \langle T_\sigma \rangle_s^\sigma]\, d\sigma + \int_{A_{\omega\eta}(t)} \mathbf{n}_{\omega\eta} \cdot ((\rho c_p)_\beta \mathbf{u}_{\beta\beta\omega} \langle T_\beta \rangle_\omega^\beta + (\rho c_p)_\beta \mathbf{u}_{\beta\sigma\omega} \langle T_\sigma \rangle_\omega^\sigma)\, dA$$

$$= - \int_{A_{\omega\eta}(t)} \mathbf{n}_{\omega\eta} \cdot (\mathbf{K}^*_{\beta\beta\omega} \cdot \nabla \langle T_\beta \rangle_\omega^\beta + \mathbf{K}^*_{\beta\sigma\omega} \cdot \nabla \langle T_\sigma \rangle_\omega^\sigma) - k_\beta \nabla \langle T_\beta \rangle_\eta^\beta]\, dA$$

$$\times \oint_C \mathbf{n}_s \cdot (\mathbf{K}^*_{\beta\beta s} \cdot \nabla_s \langle T_\beta \rangle_s^\beta + \mathbf{K}^*_{\beta\sigma s} \cdot \nabla_s \langle T_\sigma \rangle_s^\sigma)\, d\sigma - \int_{A_{\omega\eta}(t)} (a_v h)_s (\langle T_\beta \rangle_s^\beta - \langle T_\sigma \rangle_s^\sigma)\, dA. \quad \text{(A14)}$$

We can now use the surface transport theorem [40] and the surface divergence theorem [49] to place all the terms in this integral equation under the same integral sign and thus extract the differential form of the jump condition. This can be expressed in the form

$$\underbrace{(\rho c_p)_{\beta s} \frac{\partial \langle T_\beta \rangle_s^\beta}{\partial t}}_{\text{excess surface accumulation}} + \underbrace{\nabla_s \cdot [(\rho c_p)_{\beta s} \langle \mathbf{v}_\beta \rangle_s \langle T_\beta \rangle_s^\beta - (\mathbf{K}^*_{\beta\beta s} \cdot \nabla_s \langle T_\beta \rangle_s^\beta + \mathbf{K}^*_{\beta\sigma s} \cdot \nabla_s \langle T_\sigma \rangle_s^\sigma)]}_{\text{excess surface convective and conductive transport}}$$

$$- \underbrace{\nabla_s \cdot [(\rho c_p)_{\beta s}(\mathbf{u}_{\beta\beta})_s \langle T_\beta \rangle_s^\beta + (\rho c_p)_{\beta s}(\mathbf{u}_{\beta\sigma})_s \langle T_\sigma \rangle_s^\sigma]}_{\text{excess surface convective-like transport}} + \mathbf{n}_{\omega\eta} \cdot ((\rho c_p)_\beta \mathbf{u}_{\beta\beta\omega} \langle T_\beta \rangle_\omega^\beta + (\rho c_p)_\beta \mathbf{u}_{\beta\sigma\omega} \langle T_\sigma \rangle_\omega^\sigma)$$

$$- \mathbf{n}_{\omega\eta} \cdot [(\rho c_p)_\beta \langle \mathbf{v}_\beta \rangle_\omega \langle T_\beta \rangle_\omega^\beta - \mathbf{K}^*_{\beta\beta\omega} \cdot \nabla \langle T_\beta \rangle_\omega^\beta - \mathbf{K}^*_{\beta\sigma\omega} \cdot \nabla \langle T_\sigma \rangle_\omega^\sigma]$$

$$= - \mathbf{n}_{\omega\eta} \cdot [(\rho c_p)_\beta \langle \mathbf{v}_\beta \rangle_\eta \langle T_\beta \rangle_\eta^\beta - k_\beta \nabla \langle T_\beta \rangle_\eta^\beta] - \underbrace{(a_v h)_s (\langle T_\beta \rangle_s^\beta - \langle T_\sigma \rangle_s^\sigma)}_{\text{excess surface heat exchange}} \quad \text{(A15)}$$

Results can be simplified by imposing the continuity conditions

$$\text{B.C.1} \quad \langle T_\beta \rangle_\omega^\beta = \langle T_\beta \rangle_\eta^\beta \quad \text{at the } \omega\text{–}\eta \text{ boundary} \quad \text{(A16)}$$

$$\text{B.C.2} \quad \langle \mathbf{v}_\beta \rangle_\omega = \langle \mathbf{v}_\beta \rangle_\eta \quad \text{at the } \omega\text{–}\eta \text{ boundary} \quad \text{(A17)}$$

since this eliminates the convective transport term from equation (A15) and we obtain

Jump condition for the β-phase

$$\underbrace{(\rho c_{\mathrm{p}})_{\beta s}\frac{\partial\langle T_\beta\rangle_s^\beta}{\partial t}}_{\text{excess surface accumulation}}+\underbrace{\nabla_s\cdot[(\rho c_{\mathrm{p}})_{\beta s}\langle \mathbf{v}_\beta\rangle_s\langle T_\beta\rangle_s^\beta-(\mathbf{K}^*_{\beta\beta s}\cdot\nabla_s\langle T_\beta\rangle_s^\beta+\mathbf{K}^*_{\beta\sigma s}\cdot\nabla_s\langle T_\sigma\rangle_s^\sigma)]}_{\text{excess surface convective and conductive transport}}$$

$$-\underbrace{\nabla_s\cdot[(\rho c_{\mathrm{p}})_{\beta s}(\mathbf{u}_{\beta\beta})_s\langle T_\beta\rangle_s^\beta+(\rho c_{\mathrm{p}})_{\beta s}(\mathbf{u}_{\beta\sigma})_s\langle T_\sigma\rangle_s^\sigma]}_{\text{excess surface convective-like transport}}+\underbrace{\mathbf{n}_{\omega\eta}\cdot((\rho c_{\mathrm{p}})_\beta\mathbf{u}_{\beta\beta\omega}\langle T_\beta\rangle_\omega^\beta+(\rho c_{\mathrm{p}})_\beta\mathbf{u}_{\beta\sigma\omega}\langle T_\sigma\rangle_\omega^\sigma)}_{\text{normal flux of convective-like transport}}$$

$$+\mathbf{n}_{\omega\eta}\cdot(\mathbf{K}^*_{\beta\beta\omega}\cdot\nabla\langle T_\beta\rangle_\omega^\beta+\mathbf{K}^*_{\beta\sigma\omega}\cdot\nabla\langle T_\sigma\rangle_\omega^\sigma)=\mathbf{n}_{\eta\omega}\cdot(k_\beta\nabla\langle T_\beta\rangle_\eta^\beta)-\underbrace{(a_{\mathrm{v}}h)_s(\langle T_\beta\rangle_s^\beta-\langle T_\sigma\rangle_s^\sigma)}_{\text{excess surface heat exchange}}\quad\text{(A18)}$$

The analysis for the σ-phase will be identical to that which led us to equation (A18) and we simply list the result as

Jump condition for the σ-phase

$$\underbrace{(\rho c_{\mathrm{p}})_{\sigma s}\frac{\partial\langle T_\sigma\rangle_s^\sigma}{\partial t}}_{\text{excess surface accumulation}}-\underbrace{\nabla_s\cdot[(\mathbf{K}_{\sigma\beta s}\cdot\nabla_s\langle T_\beta\rangle_s^\beta+\mathbf{K}_{\sigma\sigma s}\cdot\nabla_s\langle T_\sigma\rangle_s^\sigma)]}_{\text{excess surface conductive transport}}$$

$$-\underbrace{\nabla_s\cdot[(\rho c_{\mathrm{p}})_{\sigma s}(\mathbf{u}_{\sigma\beta})_s\langle T_\beta\rangle_s^\beta+(\rho c_{\mathrm{p}})_{\sigma s}(\mathbf{u}_{\sigma\sigma})_s\langle T_\sigma\rangle_s^\sigma]}_{\text{excess surface convective-like transport}}+\underbrace{\mathbf{n}_{\omega\eta}\cdot[(\rho c_{\mathrm{p}})_\sigma\mathbf{u}_{\sigma\beta\omega}\langle T_\beta\rangle_\omega^\beta+(\rho c_{\mathrm{p}})_\sigma\mathbf{u}_{\sigma\sigma\omega}\langle T_\sigma\rangle_\omega^\sigma]}_{\text{normal flux of convective-like transport}}$$

$$+\mathbf{n}_{\omega\eta}\cdot(\mathbf{K}_{\sigma\beta\omega}\cdot\nabla\langle T_\sigma\rangle_\omega^\beta+\mathbf{K}_{\sigma\sigma\omega}\cdot\nabla\langle T_\sigma\rangle_\omega^\sigma)=\underbrace{(a_{\mathrm{v}}h)_s(\langle T_\beta\rangle_s^\beta-\langle T_\sigma\rangle_s^\sigma)}_{\text{excess surface heat exchange}}.\quad\text{(A19)}$$

Clearly these two jump conditions, in their general form, are very complicated; however, it seems likely that surface transport and the convective-like flux can be neglected in order to obtain useful forms of these two results.